

Day 20

Bayes and Kalman Filter

Combining Two Noisy Measurements

- ▶ recall from the last lecture that the minimum variance estimate for combining two noisy measurements

$$\mu = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} x_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} x_2 = x_1 + \underbrace{\frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}}_{\text{Kalman gain}} \underbrace{(x_2 - x_1)}_{\text{measurement difference}}$$

previous measurement

Kalman gain

$$\text{var}(\mu) = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

- ▶ claim: the estimate is a special case of the discrete Kalman filter algorithm

Discrete Kalman Filter

- ▶ estimates the state x of a discrete-time controlled process that is governed by the linear stochastic difference equation

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

plant model
process model

with a measurement

$$z_t = C_t x_t + \delta_t$$

measurement model
observation model

Components of a Kalman Filter

A_t

Matrix ($n \times n$) that describes how the state evolves from t to $t-1$ without controls or noise.

B_t

Matrix ($n \times 1$) that describes how the control u_t changes the state from t to $t-1$.

C_t

Matrix ($k \times n$) that describes how to map the state x_t to an observation z_t .

ε_t

Random variables representing the process and measurement noise that are assumed to be

δ_t

independent and normally distributed with covariance R_t and Q_t respectively.

Kalman Filter Algorithm

1. Algorithm **Kalman_filter**($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$):

2. Prediction:

3.
$$\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$$

4.
$$\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$$

5. Correction:

6.
$$K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

7.
$$\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$$

8.
$$\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$$

9. Return μ_t, Σ_t

Combining Two Noisy Measurements

- ▶ combining two noisy measurements of a fixed scalar quantity is a static 1D-state estimation problem
 - ▶ the state does not evolve as a function of time and does not depend on any control input

$$A_t = 1, \quad B_t = 0, \quad R_t = 0 \quad x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t \\ = x_{t-1}$$

- ▶ our measurements are direct (noisy) measurements of the state

$$C_t = 1, \quad Q_t = \sigma_t^2 \quad z_t = x_t + \delta_t$$

Combining Two Noisy Measurements

- ▶ start by initializing the Kalman filter with the first measurement and its variance

estimated
state

$$\mu_1 = x_1$$

estimated
state
covariance

$$\Sigma_1 = \sigma_1^2$$

- ▶ now substitute into the Kalman filter algorithm

Plant or Process Model

- ▶ describes how the system state changes as a function of time, control input, and noise

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

- ▶ x_t state at time t
- ▶ u_t control inputs at time t
- ▶ ε_k process noise at time t (assumed Gaussian with covariance R_t)
- ▶ A_t state transition model or matrix at time t
- ▶ B_t control-input model or matrix at time t
- ▶ note that the model is linear and assumes additive Gaussian noise

Measurement Model

- ▶ describes how sensor measurements vary as a function of the system state

$$z_t = C_t x_t + \delta_t$$

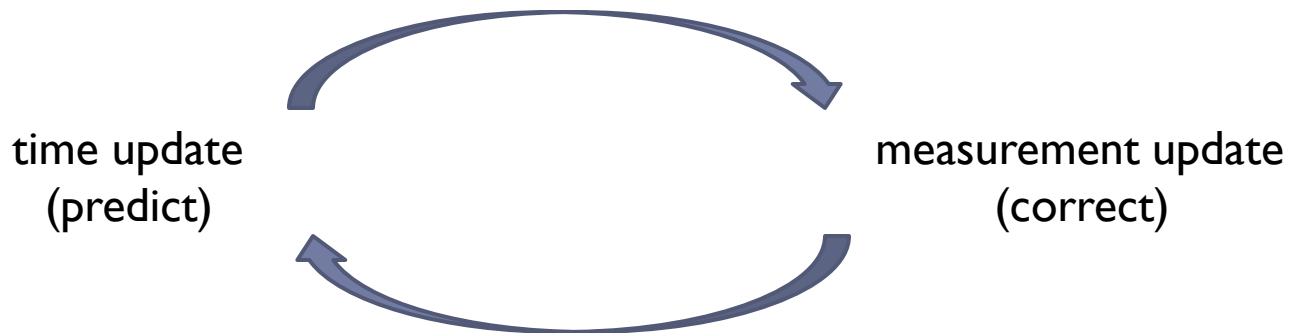
- ▶ z_t sensor measurement at time t
 - ▶ δ_t sensor noise at time t (assumed Gaussian with covariance Q_t)
 - ▶ C_t observation model or matrix
- ▶ notice that the model is linear and assumes additive Gaussian noise

Kalman Filter

- ▶ the Kalman filter is a provably optimal (in terms of least-squared error) algorithm for fusing sensor measurements to produce an estimate of the state and the state covariance
 - ▶ x_t state at time t
 - ▶ Σ_t state covariance at time t

Kalman Filter

- ▶ the Kalman filter estimates a process in two stages
 1. **prediction:** current state and state covariance estimates are projected forward in time to predict the new state and state covariance
 - ▶ “time update equations”
 2. **correction:** the sensor measurements are incorporated into the predicted state to obtain improved estimates of the state and state covariance
 - ▶ “measurement update equations”



Kalman Filter Algorithm

I. Initialization

- ▶ choose (guess) initial values for mean state and state covariance estimates

$$\mu_0$$

$$\Sigma_0$$

Kalman Filter Algorithm

2. Prediction:

- ▶ predict the next state using the plant model

$$\mu_t = A_t \mu_{t-1} + B_t u_t$$

- ▶ predicted state covariance grows (because we are not incorporating the sensor measurements yet)

$$\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$$

- ▶ R_t covariance of the plant noise

Kalman Filter Algorithm

3. **Correction:** correct the predicted state using the sensor measurement

- ▶ expected value of measurements (from measurement model)

$$\bar{z}_t = C_t \bar{\mu}_t$$

- ▶ difference between actual and expected measurements

$$r_t = z_t - \bar{z}_t$$

- ▶ measurement covariance

$$S_t = C_t \bar{\Sigma}_t C_t^T + Q_t$$

- ▶ Kalman gain

$$K_t = \bar{\Sigma}_t C_t^T S_t^{-1}$$

Kalman Filter Algorithm

4. State and state covariance:

- ▶ new state estimate incorporating most recent measurement

$$\mu_t = \bar{\mu}_t + K_t r_t$$

- ▶ new state covariance estimate

$$\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$$